Lecture 9 : Trigonometric Integrals Extra Examples

$$\int \cos^5 x dx$$

$$= \int (\cos^2 x)^2 \cos x dx = \int (1 - \sin^2 x)^2 \cos x dx$$

Let $u = \sin x$, $du = \cos x dx$

$$= \int (1-u)^2 du = \int 1 - 2u^2 + u^4 du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

$$\int \sin^3 x dx$$

$$= \int \sin^2 x \sin x dx = \int (1 - \cos^2 x) \sin x dx$$

Let $u = \sin x$ and $du = -\sin x dx$ or $-du = \sin x dx$.

$$= -\int (1-u^2) du = -[u - \frac{u^3}{3}] + C$$

$$= -[\cos x - \frac{\cos^3 x}{3}] + C = \frac{\cos^3 x}{3} - \cos x - +C$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\int \sin^4 x \cos^2 x dx$$

$$= \int (\sin^2 x) 2 \cos^2 x dx = \int [\frac{1}{2}(1 - \cos 2x)]^2 [\frac{1}{2}(1 + \cos 2x)] dx$$

$$= \frac{1}{8} \int (1 - \cos(2x))^2 (1 + \cos(2x)) dx = \frac{1}{8} \int (1 - \cos^2(2x))(1 - \cos(2x)) dx$$

you can deal with this in two ways
number 1:

$$= \frac{1}{8} \int \sin^2(2x)(1 - \cos(2x)) dx = \frac{1}{8}[\int \sin^2(2x) dx - \int \sin^2(2x)\cos(2x)) dx]$$

$$= \frac{1}{8}[\int \frac{1}{2}(1 - \cos(4x)) dx - \frac{1}{2} \int \sin^2(w) \cos(w)) dw]$$

where $w = 2x$ and $dw = 2dx$.

$$= \frac{1}{16}[x - \frac{\sin(4x)}{4}] - \frac{1}{16}\int u^2 du$$

where $u = \sin w$.

$$= \frac{1}{16}[x - \frac{\sin(4x)}{4}] - \frac{1}{16}\frac{u^3}{3}$$

$$= \frac{1}{16}[x - \frac{\sin(4x)}{4}] - \frac{1}{16}\frac{\sin^3 w}{3}$$

$$= \frac{1}{16}[x - \frac{\sin(4x)}{4}] - \frac{1}{16}\frac{\sin^3(2x)}{3}$$

Alternatively

$$\frac{1}{8}\int (1 - \cos^2(2x))(1 - \cos(2x))dx$$

$$= \frac{1}{8}\int 1 - \cos^2(2x) - \cos(2x) + \cos^3(2x)dx$$

$$= \frac{1}{8}[x - \frac{1}{2}\int (1 + \cos(4x))dx - \frac{\sin(2x)}{2} + \int \cos^2(2x)\cos(2x)dx]$$

$$= \frac{1}{8}[x - \frac{1}{2}(x + \frac{\sin(4x)}{4}) - \frac{\sin(2x)}{2} + \frac{1}{2}\int (1 - \sin^2(w))\cos(w)dw]$$

where $w = 2x$

$$= \frac{1}{8}[x - \frac{1}{2}(x + \frac{\sin(4x)}{4}) - \frac{\sin(2x)}{2} + \frac{1}{2}\int (1 - u^2)du]$$

where $u = \sin w$

$$= \frac{1}{8}[x - \frac{1}{2}(x + \frac{\sin(4x)}{4}) - \frac{\sin(2x)}{2} + \frac{1}{2}(u - \frac{u^3}{3})] + C$$

$$= \frac{1}{8}[x - \frac{1}{2}(x + \frac{\sin(4x)}{4}) - \frac{\sin(2x)}{2} + \frac{1}{2}(\sin(2x) - \frac{\sin^3(2x)}{3})] + C$$

$$= \frac{1}{16}x - \frac{\sin(4x)}{64} - \frac{\sin^3(2x)}{48} + C$$

$$\int \sin^2 x dx$$

$$= \frac{1}{2}\int (1 - \cos(2x))dx = \frac{x}{2} - \frac{1}{2}\frac{\sin(2x)}{2} + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\int \sec^4 \tan x dx$$

$$= \int \sec^2 x \tan x \sec^2 x dx$$

Let $u = \tan x$, $du = \sec^2 x dx$, $\sec^2 x = 1 + \tan^2 x$.

$$= \int (1 + u^2)udu = \int u + u^3 du = \frac{u^2}{2} + \frac{u^4}{4} + C$$

$$\frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

$$\int \sec^3 x \tan^5 x dx$$

$$= \int \sec^2 x \tan^4 x \sec x \tan x dx$$

Let $u = \sec x$, $du = \sec x \tan x dx$, $\tan^2 x = \sec^2 x - 1$.

$$= \int u^2(u^2-1)^2 du = \int u^2(u^4 - 2u^2 + 1) du = \int u^6 - 2u^4 + u^2 du$$

$$= \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

$$\int \sec^3 x \tan x dx$$

$$= \int \sec^2 x \sec x \tan x dx = \int u^2 du$$

where $u = \sec x$

$$= \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C.$$

$$\int \sec^m x \tan^n x dx$$

If m odd and n is even we can reduce to powers of secant using the identity $\sec^2 x = 1 + \tan^2 x$.

Example $\int \sec^3 x \tan^2 x dx$

$$= \int \sec^3 x(\sec^2 x - 1) dx = \int \sec^5 x - \sec^3 x dx$$

See how to deal with these below.

We have the following results for **powers of secant**

Example

$$\int \sec^0 x dx = \int 1 dx = x + C.$$

Example

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

Proof
$$\int \sec x dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
Using the substitution $u = \sec x + \tan x$, we get $du = \sec^2 x + \sec x \tan x$ giving us that the above integral is
$$\int \frac{1}{u} du = \ln |u| = \ln |\sec x + \tan x| + C.$$

Example
$$\int \sec^3 x dx$$
use integration by parts with $u = \sec x, \quad dv = \sec^2 x dx$ to get (a recurring integral)
$du = \sec x \tan x$ and $v = \tan x$
$$\int \sec^3 x dx = \sec x \tan x - \int \tan^2 x \sec x dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$
$$\sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$
Solving for $\int \sec^3 x dx$ we get
$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$
giving us
$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec^1 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

In fact for $n \geq 3$, we can derive a reduction formula for powers of sec in this way (Using Integration by parts):
$$\boxed{\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.}$$

Powers of tangent can be reduced using the formula $\tan^2 x = \sec^2 x - 1$

Example
$$\int \tan^0 x dx = \int 1 dx = x + C.$$

Example

$$\int \tan x dx = \ln|\sec x| + C$$

Proof

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

Using the substitution $u = \cos x$, we get $du = -\sin x$ giving us that the above integral is

$$\int \frac{-1}{u} du = -\ln|u| = \ln|\sec x| + C.$$

Example

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

Example

$$\int \tan^3 x dx = \int (\sec^2 x - 1) \tan x dx = \int (\sec^2 x) \tan x dx - \int \tan x dx$$

$$= \frac{\tan^2 x}{2} + \ln|\sec x| + C.$$

In fact for $n \geq 2$, we can derive a reduction formula for powers of $\tan x$ using this method (using just substitution) :

$$\boxed{\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx}$$

$$= \int \tan^2 x \tan^{n-2} x dx = \int (\sec^2 x - 1) \tan^{n-2} x dx$$

$$\int \sec^2 x \tan^{n-2} x dx - \int \tan^{n-2} x dx$$

$$= \int u^{n-2} du - \int \tan^{n-2} x dx$$

where $u = \tan x$

$$= \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

To evaluate

$$\boxed{\int \sin(mx)\cos(nx)dx \qquad \int \sin(mx)\sin(nx)dx \qquad \int \cos(mx)\cos(nx)dx}$$

we reverse the identities

$$\sin((m-n)x) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$

$$\sin((m+n)x) = \sin mx \cos nx + \sin nx \cos mx$$
$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(nx)\sin(mx)$$
$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(nx)\sin(mx)$$

to get

$$\sin(mx)\cos(nx) = \frac{1}{2}\big[\sin((m-n)x) + \sin((m+n)x\big]$$
$$\sin(mx)\sin(nx) = \frac{1}{2}\big[\cos((m-n)x) - \cos((m+n)x\big]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\big[\cos((m-n)x) + \cos((m+n)x\big]$$

Example $\int \sin 7x \cos 3x dx$

$$= \int \frac{1}{2}\big[\sin(4x) + \sin(10x)\big]dx$$
$$= \frac{1}{2}[\frac{-\cos(4x)}{4} + \frac{-\cos(10x)}{10}$$
$$= \frac{-\cos(4x)}{8} - \frac{\cos(10x)}{20}$$

$$\int \cos 8x \cos 2x dx$$

$$= \frac{1}{2}\int \big[\cos(6x) + \cos(10x\big]dx$$
$$= \frac{1}{2}[\frac{\sin(6x)}{6} + \frac{\sin(10x)}{10}]$$
$$= \frac{\sin(6x)}{12} + \frac{\sin(10x)}{20}$$

$$\int \sin x \sin 2x dx$$

$$= \frac{1}{2}\int \big[\cos(-x) - \cos(3x)\big]dx$$
$$= \frac{1}{2}[\frac{\sin(-x)}{-1} - \frac{\sin(3x)}{3}]$$
$$= \frac{-\sin(-x)}{2} - \frac{\sin(3x)}{6}$$